Announcements

1) Exam next Thursday
will not cover 3.2
(mean value theorem)

2) Monday is Halloween.
Costumes = Candy.

3) Monday, Practice
Exam, Tuesday review.

4) Amanda's Review Session on Tuesday, time TBD.

How to tell whether a point is a local max or min?

1st Derivative Test

Find all points where f' is zero or does not exist. Plot these points on a number line, find intervals of increase/decrease for f.

If the intervals change from increasing to decreasing (and f is defined at the point where the behavior changes) then f has a local max.

If the intervals change from decreasing to increasing (and f is defined at the point where the behavior changes) then f has a local min.

This test never fails,
but it is a bit long.

There is an alternative
that sometimes fails.

2nd Derivative Test

Suppose $f'(c) = 0$ for some number c.

i) If $f''(c) < 0$, then f has a local max at $x = c$.

ii) If $f''(c) > 0$, then f has a local min at $x = c$.

iii) If $f''(c) = 0$, the test fails and you know nothing.

Examples where $f''(c) = 0$

a) $f(x) = x^4$. $f'(0) = f''(0) = 0$

and f has a local (absolute) minimum at $x = 0$.

b) $f(x) = -x^4$. $f'(0) = f''(0) = 0$

and f has a local (absolute) max at $x = 0$.

c) $f(x) = x^3$. $f'(0) = f''(0) = 0$.

f has neither a local max nor a local min at $x = 0$.

Example 1. $g(x) = \dfrac{2x-5}{x^2+6}$

Find all local maxima and minima.

Use quotient rule.

$$g'(x) = \frac{(x^2+6)(2) - (2x-5)(2x)}{(x^2+6)^2}$$

$$= \frac{2x^2+12-4x^2+10x}{(x^2+6)^2}$$

$$g'(x) = \frac{-2x^2+10x+12}{(x^2+6)^2}$$

$g'(x)=0$ when numerator is 0,

$g'(x)$ undefined when denominator is undefined.

$g'(x)=0$ $-2x^2+10x+12=0$

$$-2(x^2-5x-6)=0$$

$$-2(x-6)(x+1)=0$$

$$\boxed{x=6,-1}$$

$g'(x)$ undefined

$$(x^2+6)^2=0$$

$$x^2+6=0$$

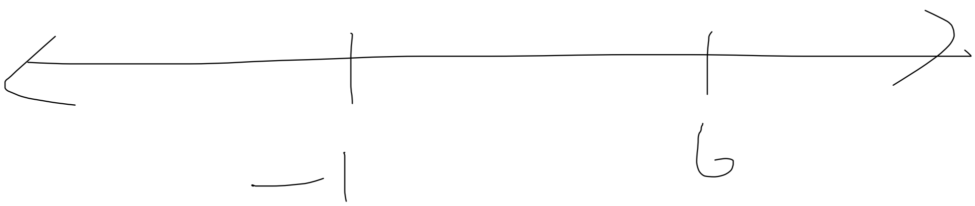
$x^2=-6$, g' is

then defined for all real

numbers.

Points are $x=6$, $x=-1$.

Plot on number line.



Intervals $(-\infty,-1)$, $(-1,6)$, $(6,\infty)$. Plug points from these intervals in g'.

$(-1, 6)$ $\quad x = 0$

$$g'(0) = \frac{-2(-6)(1)}{6^2} > 0$$

so g is increasing

$(6, \infty)$ $\quad x = 7$

$$g'(7) = \frac{-2(7-6)(7+1)}{55^2} < 0$$

so g is decreasing

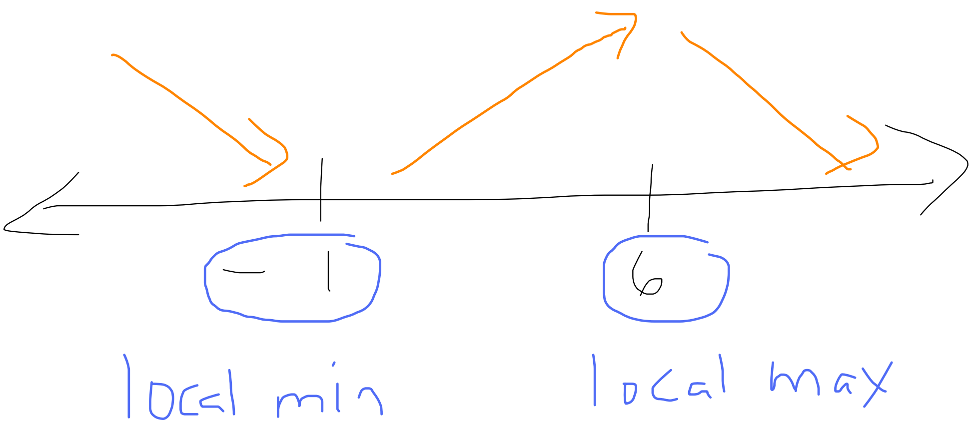
$(-\infty, -1)$ $\quad x = -2$

$$g'(-2) = \frac{-2(-2-6)(-2+1)}{10^2}$$

$$< 0$$

so g is decreasing.

Number line



2^{nd} derivative Test

$$g'(x) = \frac{-2x^2+10x+12}{(x^2+6)^2}$$

points we care about: $x = 6, -1$

$$g''(x) =$$

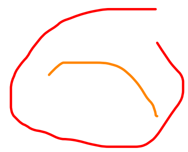
$$\frac{(x^2+6)^2(-4x+10) - (-2x^2+10x+12)(2)(2x)(x^2+6)}{(x^2+6)^4}$$

plug in $x = 6, -1$.

you'll get $g''(6) < 0$, $g''(-1) > 0$.

Concavity, Inflection Points, the Second Derivative

Definition: f is concave up  (or simply concave) on an interval I if all tangent lines to f stay below its graph on I.

Similarly, f is concave down  (or simply convex) on I if all tangent lines to the graph are above the graph on I.

Relation of Second Derivative to Concavity

If $f''(x) > 0$ on I, then f is concave up on I.

If $f''(x) < 0$ on I, then f is concave down on I.

Definition. (inflection points)

Inflection points for f are points where the concavity changes

Usually where f'' is zero or does not exist.

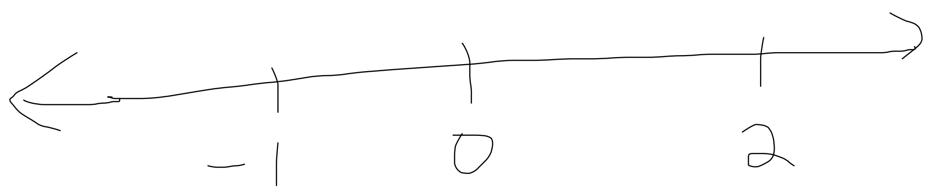
Example 2:

$$f(x) = 3x^4 - 4x^3 - 12x^2 - 15$$

Find all intervals of increase / decrease, local maxima and minima, inflection points, and intervals of concavity.

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x-2)(x+1)$$

Points where $f'(x) = 0$ are $x = 0, -1, 2$.



Plug in points to f'.

$(-\infty, -1)$ $\quad x = -2$

$$f'(-2) = 12(-2)(-4)(-1)$$

$$< 0$$

f is decreasing

$(-1, 0)$ $\quad x = -\frac{1}{2}$

$$f'\left(-\frac{1}{2}\right) = 12\left(-\frac{1}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{1}{2}\right)$$

$$> 0$$

f is increasing

$(0,2)$ $x=1$

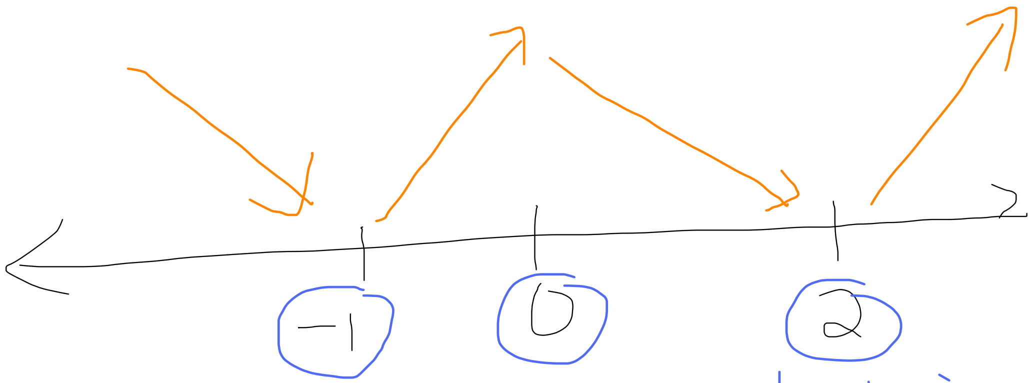
$f'(1)=12(1)(-1)(2)<0$

f is decreasing

$(2,\infty)$ $x=3$

$f'(3)=12(3)(1)(4)>0$

f is increasing



local min local max local min

$$f''(x) = 36x^2 - 24x - 24$$

$$= 12(3x^2 - 2x - 2)$$

$$0 = f''(x) = 12(3x^2 - 2x - 2)$$

There's no easy factorization

Use quadratic formula

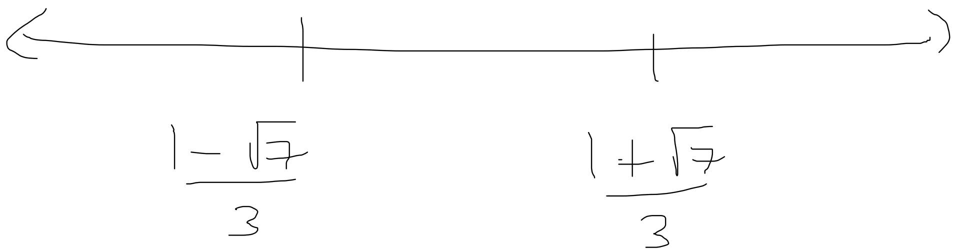
$$3x^2 - 2x - 2 = 0$$

$$x = \frac{2 \pm \sqrt{4+24}}{6}$$

$$= \frac{2 \pm 2\sqrt{7}}{6}$$

$$= \frac{1 \pm \sqrt{7}}{3}$$

Plot on a number line



Pick points in the intervals, plug into f''.

$$\left(-\infty, \frac{1-\sqrt{7}}{3}\right) \qquad x=-1$$

$$f''(-1) = 12(3(1)+2-2)$$

$$>0$$

so f is concave up.

$$\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right) \qquad x=0$$

$$f''(0) = 12(-2)$$

so f is concave down.

$\left(\frac{1+\sqrt{7}}{3}, \infty\right)$ $\quad x = 3$

$$f''(3) = 12\left(3(3)^2 - 2(3) - 2\right)$$

$$> 0$$

so f is concave up

$\frac{1-\sqrt{7}}{3}$ $\qquad$ $\frac{1+\sqrt{7}}{3}$

inflection points

Summarizing: f is

increasing on $(-1, 0)$ and

$(2, \infty)$; decreasing on

$(-\infty, -1)$ and $(0, 2)$.

f is concave up on

$\left(-\infty, \frac{1-\sqrt{7}}{3}\right)$ and $\left(\frac{1+\sqrt{7}}{3}, \infty\right)$,

concave down on $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right)$.

Local Maxima at $x = 0$

Local Minima at $x = -1, 2$

Inflection Points at $x = \frac{1 \pm \sqrt{7}}{3}$